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On Large Isolated Regions in Supercritical Percolation

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We consider supercritical vertex percolation in $\mathbb{Z}^d$ with any non-degenerate uniform oriented pattern of connection. In particular, our results apply to the more special unoriented case. We estimate the probability that a large region is isolated from ∞. In particular, "pancakes" with a radius $r \to \infty$ and constant thickness, parallel to a constant linear subspace L, are isolated with probability, whose logarithm grows asymptotically as $\asymp r^{\dim(L)}$ if percolation is possible across L and as $\asymp r^{\dim(L)-1}$ otherwise. Also we estimate probabilities of large deviations in invariant measures of some cellular automata.

KEY WORDS: Oriented percolation; large deviations; connected sets; Peierls contour estimations; cellular automata; invariant measures.

We consider a d-dimensional real space $\mathbb{R}^d$ with a basis $(e_1,..., e_d)$, which includes a d-dimensional integer space

$$\mathbb{Z}^d = \left\{ v = \sum_{i=1}^{d} v_i e_i,\ v_i \in \mathbb{Z} \right\}.$$

In both spaces we use the norm $\|v\| = \max_i |v_i|$. Elements of $\mathbb{R}^d$ are called points or vectors, elements of $\mathbb{Z}^d$ are called vertices or integer vectors. The paper consists of three parts. In the first two parts we prove two statements about percolation. Similar estimations in another setting were obtained in one direction in ref. 1 and in the other in ref. 2. In the third part we derive a conclusion about cellular automata.

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PART I. THE STANDARD PERCOLATION PATTERN

A *finite path* from v to w in an oriented or non-oriented graph is a finite sequence "vertex-edge-vertex-edge-$\cdots$-vertex," where the first vertex is v, the last vertex is w and every edge goes from the previous to the next vertex. The length of a finite path is the number of edges in it. An *infinite path* starting at v is an infinite sequence "vertex-edge-vertex-edge-$\cdots$," where the first vertex is v. A *bi-infinite path* is a bi-infinite sequence "$\cdots$-edge-vertex-edge-vertex-edge-$\cdots$." A path is called *self-avoiding* if all the vertices in its sequence are different. We say that a path *goes to* ∞ if it is infinite and self-avoiding. A path *belongs* to a set S if all of its vertices belong to S. A path *avoids* a set S if none of its vertices belongs to S, otherwise it *hits* S. We say that a path starts at a set if it starts at some element of this set.

Throughout Parts I and II we assume that every element of $\mathbb{Z}^d$ is either closed with probability ε or open with probability $1-\varepsilon$ independently of others. In both parts there is a finite set of non-zero vectors $\{n_1,\ldots, n_\nu\} \subset \mathbb{Z}^d$, called *neighbor vectors*, and we study percolation on an oriented graph N having $\mathbb{Z}^d$ as the set of vertices, in which oriented edges go from any vertex v to the vertices $v+n_1,\ldots, v+n_\nu$, which are called *N-neighbors* of v. A finite or infinite sequence of elements of $\mathbb{Z}^d$ is called *N-path* if every of its element except the first one is a N-neighbor of the preceding term. A N-path is called *open* if all its vertices are open. We say that a vertex w is *reachable* from a vertex v if there is a finite open path from v to w and that a vertex is reachable from a set if it is reachable from some element of this set. We say that a vertex is *cut* from ∞ if the set of vertices reachable from it is finite. It is well-known that a vertex is not cut from ∞ if and only if there is an open path from this vertex to ∞. A finite set of vertices is termed *cut* from ∞ if all its elements are cut from ∞ or, which is the same, the set of vertices reachable from this set is finite. For any finite $S \subset \mathbb{Z}^d$ we denote $P_{\text{cut}}(S)$ the probability that S is cut from ∞. Given two sets $S, B \subset \mathbb{Z}^d$, we denote $W(S \mid B)$ the set of vertices reachable from S if B is the set of closed vertices. We say that B is a *barrier* for S if $W(S \mid B)$ is finite. In particular, any set in $\mathbb{Z}^d$ is its own barrier. (This is different from the approach used in some other articles, which assume that $S \cap B = \varnothing$. Our approach is designed to be applicable to cellular automata.) Thus for any finite $S \subset \mathbb{Z}^d$ there is a finite barrier, so we may denote $k_0(S)$ the smallest number of elements in a barrier for S.

In Part I we consider the special case

$$\{n_1,\ldots, n_\nu\} = \{e_1,\ldots, e_d\} \tag{1}$$

and we use a non-oriented graph D with $\mathbb{Z}^d$ as the set of vertices, in which any $v, w \in \mathbb{Z}^d$ are connected with an edge if $\|v-w\|=1$, in which case v and w are called *D-neighbors*. Edges of D are called *D-edges*. A sequence of elements of $\mathbb{Z}^d$ is called a *D-path* if every of its two next elements are D-neighbors. A set $S \subseteq \mathbb{Z}^d$ is called *D-connected* if for any two of its elements there is a D-path in S, connecting them.

Theorem 1. Assuming (1), for every $d \geqslant 2$ there are $\varepsilon_d^* > 0$ and $C_d > 0$ such that for all D-connected $S \subset \mathbb{Z}^d$ and all $\varepsilon \in (0, \varepsilon_d^*)$

$$\varepsilon^{k_0(S)} \leqslant P_{\text{cut}}(S) \leqslant (C_d \cdot \varepsilon)^{k_0(S)}.$$

Proof of Theorem 1. Let $|\cdot|$ denote cardinality. *The lower estimation* is trivial: since there is a barrier B of S, such that $|B| = k_0(S)$, the probability that all its elements are closed equals $\varepsilon^{k_0(S)}$, whence the probability that S is cut from ∞ cannot be less than this. The bulk of Part I is a proof of *the upper estimation* of Theorem 1. A barrier for S is called *minimal* if all its proper subsets are not barriers for S. For any $S \subset \mathbb{Z}^d$ and $v \in \mathbb{Z}^d$ we call a *shift* of S by a vector v the set $S+v = v+S = \{s+v, s \in S\}$. Also for any $S_1, S_2 \subset \mathbb{R}^d$ we denote $S_1 + S_2 = \{a+b, a \in S_1, b \in S_2\}$.

Lemma 1. If $S \subset \mathbb{Z}^d$ is finite and B is a minimal barrier for S, then any shift of B by a non-zero vector is not a minimal barrier for S.

Proof. For any finite set $S \subset \mathbb{Z}^d$ and any $i \in \{1,..., d\}$ let us denote $m_i(S)$ the minimum of ith components of elements of S. Let us prove by contradiction that if B is a minimal barrier for a finite set $S \subset \mathbb{Z}^d$, then $m_i(B) = m_i(S)$ for all i. *First, let* $m_i(B) > m_i(S)$. Then B is not a barrier for S because S contains at least one element v, whose ith component is less than $m_i(B)$ and therefore all the vertices of the form $v + w_1 + \cdots + w_n$, where n is any natural number and all w_k are e_j with $j \neq i$, are reachable from v. *Now let* $m_i(B) < m_i(S)$. Then, even if B is a barrier for S, it is not minimal, because it remains a barrier, when we exclude from it all the vertices, whose ith component is less than $m_i(S)$. *Now let* a minimal barrier for S be shifted by a non-zero vector. Then m_i of the shift is different from $m_i(S)$ for at least one i, whence the shift cannot be a minimal barrier for S. ∎

Lemma 2. For any finite $S \subset \mathbb{Z}^d$ and any $B \subset \mathbb{Z}^d$:

(a) $S \subseteq B \cup W(S \mid B)$;

(b) If B is a minimal barrier for S, then $B \subseteq S \cup (W(S \mid B) + N)$;

(c) Any minimal barrier for S is finite;

(d) Any barrier for S contains a minimal barrier for S.

Proof of (a). The proof of (a) is evident.

Proof of (b). Let us denote $C = W(S \mid B) + N$ (the set of N-neighbors of elements of $W(S \mid B)$), take any $b \in B$ and prove that $b \in S \cup C$. Let us denote $B' = B \setminus \{b\}$. Since B is a minimal barrier, B' is not a barrier, whence there is a N-path from S to ∞, which avoids B'. Since B is a barrier, this path must contain b and it is the only term of our path, which belongs to B. Let us denote a the last term of this path, which belongs to S. If $a = b$, then $b \in S$ and we are done. So let $a \neq b$. Then a has to precede b in our path because otherwise there would be a path from $a \in S$ to ∞ avoiding B, whence B would not be a barrier of S. Notice that a does not belong to B, because if it would, it would belong to B', but our path avoids B'. Notice that $S \subseteq B \cup W(S \mid B)$, whence $a \in W(S \mid B)$. Then we prove by induction that all the terms in this path, which are between a and b, also belong to $W(S \mid B)$, whence $b \in W(S \mid B) + N$.

Proof of (c). Since B is a barrier of S, the set $W(S \mid B)$ is finite. Therefore C defined in the proof of (b) is finite, so from item (b) B is finite also.

Proof of (d). Given any barrier B, let us denote $B' = B \cap (S \cup C)$, where C was defined in the proof of (a). B' is finite because S and C are finite. Let us prove that B' is a barrier. If it is not, there is an infinite self-avoiding N-path starting at some $a \in S$ and avoiding B'. So $a \notin B$, whence $a \in W(S \mid B)$ from item (a). Since B is a barrier, $W(S \mid B)$ is finite, so there is the last term in our path belonging to $W(S \mid B)$. The next term, which we denote c, belongs to C. Then $c \notin B$ because our path avoids $B' = B \cap (S \cup C)$. But then $c \in W(S \mid B)$, which contradicts our choice of the preceding term. So B' is a barrier. Since B' is finite, we have proved that any barrier contains a finite barrier. If it is not minimal, we exclude from it elements one after another until we get a minimal barrier. ▮

Lemma 3. For any finite $S \subset \mathbb{Z}^d$

$$P_{\text{cut}}(S) \leqslant \sum_B \varepsilon^{|B|} = \sum_{k=k_0}^{\infty} M_k \cdot \varepsilon^k,$$

where the first sum is taken over all minimal barriers B for S. In the second sum $M_k = M_k(S)$ is the number of minimal barriers for S containing k elements and $k_0 = k_0(S)$ is the same as before, i.e., the minimal value of k for which $M_k > 0$.

Proof. Due to item (d) of Lemma 2, any configuration, where S is cut from ∞, belongs to a cylinder set defined by a condition "all vertices in B are closed," where B is a minimal barrier for S. Therefore the event "S is cut from ∞" is covered by the events "all elements of B are closed," where B runs all minimal barriers of S, all of which are finite from item (c) of Lemma 2. ▮

The sum in Lemma 3 is of the typical form in a Peierls argument and its estimation should be based on an exponential estimation of M_k, which becomes our main goal in Part I. We also use a non-oriented graph E with $\mathbb{Z}^d$ as the set of vertices, in which any $v \in \mathbb{Z}^d$ is connected with vertices $v \pm e_i$ for all $i = 1,..., d$. Edges of E are called *E-edges*. Thus graphs M, D and E have one and the same set $\mathbb{Z}^d$ of vertices. In Part I, due to (1), N and E have one and the same set of edges, only in N they are oriented and in E they are not. A sequence of elements of $\mathbb{Z}^d$ is called a *E-path* if every of its two next elements are E-neighbors. A set $S \subseteq \mathbb{Z}^d$ is called *E-connected* if for any two of its elements there is an E-path in S, connecting them. In Part I, any N-path is an E-path. If two vertices are E-neighbors, they are D-neighbors, whence any E-path is a D-path and any E-connected set is D-connected. Let us call an E-path connecting $v, w \in \mathbb{Z}^d$ *shortest* if its length equals $\sum_i |v-w|$, which is the minimum of lengths of all E-paths connecting v with w.

Lemma 4. If $v, w \in \mathbb{Z}^d$ are D-neighbors, then any two different terms of a shortest E-path connecting v with w, are D-neighbors.

Proof. Suppose that $v, w \in \mathbb{Z}^d$ are D-neighbors. For every $i \in \{1,..., d\}$, if p is a term of a shortest path connecting v with w, then p_i equals v_i or w_i. Therefore, if p and q are two terms of such a path, then $|p_i - q_i| \leqslant |v_i - w_i| \leqslant 1$. ▮

For any $S \subset \mathbb{Z}^d$ we denote $\bar{S}$ the set of those elements of $\mathbb{Z}^d$, from which there is no D-path to ∞ avoiding S.

Lemma 5. For any $S \subset \mathbb{Z}^d$:

(a) $S \subseteq \bar{S}$;

(b) If S is finite, then $\bar{S}$ is finite also.

Proof of (a). Any path, starting at a vertex $v \in S$, includes v and therefore does not avoid S, whence $v \in \bar{S}$.

Proof of (b). Let us break $\bar{S} \setminus S$ into classes of equivalence, called *clusters*, two elements of $\bar{S} \setminus S$ being equivalent if they are connected by a D-path in $\bar{S} \setminus S$. The set of clusters is finite because we can estimate their number as follows: on one hand, every cluster has at least one D-neighbor belonging to S, on the other hand every element of S (in fact, every element of $\mathbb{Z}^d$) has 3^d-1 D-neighbors. Therefore the number of clusters does not exceed $(3^d-1)\cdot|S|$. All the clusters are finite, because if one of them were infinite, there would be a path from any of its elements to ∞ in this cluster, which contradicts the definition of $\bar{S}$. So the union of clusters is finite also. ∎

Lemma 6. If $S\subset\mathbb{Z}^d$ is finite and D-connected, then for any B barrier of S the set $(S\cap B)\cup W(S\mid B)$ is finite and D-connected.

Proof. Let us denote $C=(S\cap B)\cup W(S\mid B)$. Since B is a barrier of S, the set $W(S\mid B)$ is finite, whence C is finite. Let us prove that C is D-connected. Let us take any $a, b\in C$ and consider three cases.

First Case: $a, b\in S\cap B$. Since S is D-connected, there is a D-path in S connecting a with b. From item (a) of Lemma 2, this D-path belongs to C.

Second Case: $a, b\in W(S\mid B)$. Since any $v\in W(S\mid B)$ is reachable from some $w\in S$, there is a N-path from w to v, all of whose terms are also reachable from w and therefore belong to $W(S\mid B)$. Since any N-path is a D-path, we can connect any two elements of $W(S\mid B)$ by D-paths belonging to $W(S\mid B)$ with some elements of S. Since S is D-connected, these two elements are connected with a D-path belonging to S. Then we take a concatenation of these three paths and, arguing like in the first case, we prove that this path belongs to C.

Third Case: $a\in B\cap S$ and $b\in W(S\mid B)$. This case can be treated like the second one. ∎

Now we are approaching our main task: to estimate M_k. To do this, we need to prove that any minimal barrier B can be included in a connected set, whose cardinality is $O(|B|)$. Informally speaking, our proof is based on the idea that the boundary of a bounded connected set is also connected. Following p. 138 of ref. 3, for any $S\subset\mathbb{Z}^d$ we denote $\partial_{\text{ext}}S$ the set of those elements of $\mathbb{Z}^d\setminus S$, which have a D-neighbor in S and from which there is a D-path to ∞, avoiding S. Also for any $S\subset\mathbb{Z}^d$ let us denote $\partial_D S$ the set of those elements of $\mathbb{Z}^d\setminus S$, which have a D-neighbor in S.

Lemma 7. For any finite $S\subset\mathbb{Z}^d$ the sets $\partial_{\text{ext}}S$ and $\partial_D\bar{S}$ coincide.

Proof. From item (a) of Lemma 5, $\partial_{\text{ext}}S\subseteq\partial_D\bar{S}$. It remains to prove that $\partial_D\bar{S}\subseteq\partial_{\text{ext}}S$. Take any $v\in\partial_D\bar{S}$. By definition of ∂_D, v belongs to $\mathbb{Z}^d\setminus\bar{S}$ and has a D-neighbor $w\in\bar{S}$. Let us take any shortest path $p_0=v,\ldots, p_m=w$

from v to w. Let p_k be the first term in this path belonging to $\bar{S}$. If p_k did not belong to S, we could start at this term, go back on the path to v and then to ∞ avoiding S, whence p_k would not belong to $\bar{S}$. So $p_k \in S$. From Lemma 4, v and p_k are D-neighbors, whence v has a D-neighbor in S. Thus $v \in \partial_{\text{ext}} S$. ∎

Lemma 8. If $d \geqslant 2$, then for any finite D-connected set $S \subset \mathbb{Z}^d$, the set $\partial_{\text{ext}} S$ is E-connected.

Proof. This lemma is proved as Lemma 2.23 on p. 139 of ref. 3. Lemma 2.1 in ref. 4, Lemmas 4.1 and 5.1 in ref. 5 and Proposition 6 in ref. 6 are similar and could also be used for our purpose, but not so easily.

Lemma 9. For any finite $S \subset \mathbb{Z}^d$ and any minimal barrier B of S:

(a) $B \cap \overline{W(S \mid B)} = \varnothing$;

(b) $B \subseteq S \cup \partial_{\text{ext}} W(S \mid B)$.

Proof of (a). Let us take any $b \in B$ and prove that there is a D-path from b to ∞ avoiding $W(S \mid B)$. Since B is minimal, $B \setminus \{b\}$ is not a barrier, so there is a N-path from S to ∞ avoiding $B \setminus \{b\}$. Since B is a barrier, this N-path must contain b. This path cannot pass through any element $v \in W(S \mid B)$ after it has passed through b, because there would then exist a path from S to v and then from v to ∞ avoiding B. So there is an open infinite self-avoiding N-path, hence an E-path, hence a D-path starting at b and avoiding $W(S \mid B)$.

Proof of (b). Let us take any $b \in B \setminus S$. From item (a), b does not belong to $\overline{W(S \mid B)}$. From item (b) of Lemma 2, b is a N-neighbor, hence an E-neighbor, hence a D-neighbor of some element of $W(S \mid B)$. Thus $b \in \partial_{\text{ext}} W(S \mid B)$ by definition of ∂_{ext}. ∎

Lemma 10. For any finite D-connected $S \subset \mathbb{Z}^d$ and a minimal barrier B of S the set $(S \cap B) \cup \partial_{\text{ext}} W(S \mid B)$ is finite and D-connected.

Proof. Let us denote

$$C = (S \cap B) \cup \partial_{\text{ext}} W(S \mid B) \qquad \text{and} \qquad C' = (S \cap B) \cup W(S \mid B).$$

Since B is a barrier of S, the set $W(S \mid B)$ is finite. Then from item (b) of Lemma 5, C is finite. It remains to prove that C is D-connected. First let us prove that any $v, w \in C$ are connected with a D-path in $C \cup C'$. If both v and w belong to $S \cap B$, this follows from Lemma 6. If $v \in \partial_{\text{ext}} W(S \mid B)$

and $w \in S \cap B$, then by definition of $\partial_{\exp}$, vertex v has a D-neighbor $v' \in W(S \mid B)$, which by Lemma 6 is connected with w by a D-path in C', whence v is connected with w by a D-path in $C \cup C'$. If both v and w belong to $\partial_{\text{ext}} W(S \mid B)$, they have D-neighbors $v', w' \in W(S \mid B)$, which are connected with a D-path in C' by Lemma 6, whence v and w are connected with a D-path in $C \cup C'$.

Now let us argue by induction. Notice that $C' \setminus C = W(S \mid B)$. The induction step is the following: if a D-path in $C \cup C'$ connecting $v, w \in C$ contains at least one element of $C' \setminus C = W(S \mid B)$, then there is a D-path in $C \cup C'$ connecting v, w containing a smaller number of elements of $C' \setminus C$. Let us denote our path $(p_1, \ldots, p_n)$, where $p_1 = v$ and $p_n = w$. From definition of ∂_{ext} and item (a) of Lemma 9, $C \cap \overline{W(S \mid B)} = \varnothing$ whence $v, w \notin \overline{W(S \mid B)}$. Let us break $C' \setminus C = W(S \mid B)$ into classes of equivalence, which we call *clusters* and denote $Q_1, \ldots, Q_n$, claiming two elements of $W(S \mid B)$ equivalent if there is a D-path in $W(S \mid B)$ connecting them. Let j be the smallest index such that $p_j \in W(S \mid B)$ and let Q_i be the cluster containing p_j. Let k be the greatest index such that $p_k \in Q_i$. Let us prove that $\partial_{\text{ext}} Q_i \subseteq \partial_{\text{ext}} W(S \mid B)$. By definition of ∂_{ext}, any element of $\partial_{\text{ext}} Q_i$ has a D-neighbor in $Q_i \subseteq W(S \mid B)$. It remains to prove that from any element of $\partial_{\text{ext}} Q_i$ there is a D-path to ∞ avoiding $W(S \mid B)$. Indeed, there is a D-path from p_{j-1} to ∞ avoiding $W(S \mid B)$ and therefore Q_i, namely first back along our path to $p_1 = v$ and then to ∞ using the fact that $v \notin \overline{W(S \mid B)}$. Now let us prove that

$$\partial_{\text{ext}} Q_i \subseteq \partial_{\text{ext}} W(S \mid B). \tag{2}$$

Notice that $\partial_{\text{ext}} Q_i$ does not intersect $W(S \mid B)$ by definition of ∂_{ext} and definition of clusters. Let us take any $v \in \partial_{\text{ext}} Q_i$. By definition of ∂_{ext}, vertex v has a D-neighbor in $Q_i \subseteq W(S \mid B)$. From Lemma 8, $\partial_{\text{ext}} Q_i$ is E-connected, hence D-connected. So there is a D-path from v to p_{j-1} in $\partial_{\text{ext}} Q_i$, hence avoiding $W(S \mid B)$. Also, as we have seen, there is a D-path from p_{j-1} to ∞ avoiding $W(S \mid B)$. Concatenating these paths, we obtain a D-path from v to ∞ avoiding $W(S \mid B)$. Thus (2) is proved. In particular, we have proved that $p_{j-1}, p_{k+1} \in \partial_{\text{ext}} Q_i$. From Lemma 8, the set $\partial_{\text{ext}} Q_i$ is E-connected, hence D-connected, so p_{j-1} and p_{k+1} are connected with a D-path within $\partial_{\text{ext}} Q_i \subseteq \partial_{\text{ext}} W(S \mid B) \subseteq C$, which we substitute into our path instead of $(p_j, \ldots, p_k)$. Thus we obtain another D-path in $C \cup C'$ connecting v with w, whose number of terms belonging to $C' \setminus C$ is less than in the original path. After several such steps we obtain a D-path in C connecting v with w. ∎

Now we need to prove that $|\partial_{\text{ext}} W(S \mid B)| \leqslant \text{const} \cdot |B|$. Given a finite $S \subset \mathbb{Z}^d$, for all $i \in \{1, \ldots, d\}$, we denote:

(a) $\partial_+ S$ is the set of those edges of the graph N, which lead from an element of S to an element of $\mathbb{Z}^d \setminus S$;

(b) $\partial_- S$ is the set of those edges of the graph N, which lead from an element of $\mathbb{Z}^d \setminus S$ to an element of S;

(c) $\partial_1 S = \partial_+ S \cup \partial_- S$.

Lemma 11. For any finite $S \subset \mathbb{Z}^d$:

(a) $\partial_+ S \cap \partial_- S = \varnothing$;

(b) $|\partial_+ S| = |\partial_- S|$.

Proof of (a). The proof of (a) is evident.

Proof of (b). For every $i \in \{1,\ldots, d\}$ let us denote $\partial_{+i} S$ the set of those edges belonging to ∂_+, which are parallel to the ith axis and $\partial_{-i} S$ the set of those edges belonging to ∂_-, which are parallel to the ith axis. First let us prove that for any finite $S \subset \mathbb{Z}^d$ and any $i \in \{1,\ldots, d\}$

$$|\partial_{+i} S| = |\partial_{-i} S|. \tag{3}$$

Let us concentrate on one line parallel to the ith axis. Following this line from $-\infty$ to ∞, we actually follow a bi-infinite N-path. Since S is finite, the number of N-edges from $\mathbb{Z}^d \setminus S$ to S equals the number of N-edges from S to $\mathbb{Z}^d \setminus S$ in this N-path. But elements of $\partial_{+i} S$ and $\partial_{-i} S$ belonging to this line are exactly these N-edges, so their numbers on this line are equal. Summing over all lines parallel to the ith axis, we prove (3). Now, summing (3) over $i \in \{1,\ldots, d\}$, we prove Lemma 11.

Lemma 12. For any finite $S \subset \mathbb{Z}^d$

$$|\partial_D S| \leqslant 4^d \cdot |\partial_+ S|.$$

Proof. Let us say that a vertex v is a *vassal* of an E-edge if both ends of this E-edge are D-neighbors of v. Notice that any E-edge has $2 \cdot 3^{d-1}$ vassals. Let us prove that any element of $\partial_D S$ is a vassal of some element of $\partial_1 S$. From the definition, every $v \in \partial_D S$ has a D-neighbor $w \in S$. Let us take a shortest path $p_0 = v,\ldots, p_k = w$ connecting v with w. Let p_j be the last term in this path, which does not belong to S. Then v is a vassal of the E-edge connecting p_j and p_{j+1}. Now using Lemma 11

$$|\partial_D S| \leqslant 2 \cdot 3^{d-1} \cdot |\partial_1 S| = 4 \cdot 3^{d-1} \cdot |\partial_+ S| < 4^d \cdot |\partial_+ S|. \quad \blacksquare$$

Lemma 13. If $S \subset \mathbb{Z}^d$ is finite and B is a minimal barrier of S, then

$$|\partial_{\text{ext}} W(S \mid B)| \leqslant d \cdot 4^d \cdot |B|.$$

Proof. First we use Lemma 7 and then, since $W(S \mid B)$ is finite, we can use Lemma 12 to obtain

$$|\partial_{\text{ext}} W(S \mid B)| = |\partial_D \overline{W(S \mid B)}| \leqslant 4^d \cdot |\partial_+ \overline{W(S \mid B)}|.$$

For any N-edge, parallel to the ith axis, let us call that end, where the value of the ith coordinate is greater, *positive*, and the other one *negative*. Observe that the positive end of any element of $\partial_+ \overline{W(S \mid B)}$ belongs to B, because otherwise this end would have to belong to $W(S \mid B)$, since the negative end belongs to it. On the other hand, any element of $\mathbb{Z}^d$ is a positive end of d N-edges, whence any element of B is a positive end of at most d edges belonging to $\partial_+ \overline{W(S \mid B)}$. Therefore

$$|\partial_+ \overline{W(S \mid B)}| \leqslant d \cdot |B|.$$

The last two inequalities prove Lemma 13.

Lemma 14. For any $d \geqslant 2$ there is a number C such that for any D-connected set $S \subset \mathbb{Z}^d$ and any natural k the number of minimal barriers for S, containing k vertices, does not exceed C^k.

Proof. Let $S \subset \mathbb{Z}^d$ be D-connected. For any B minimal barrier of S we imagine a non-oriented graph, whose vertices are the elements of $(S \cap B) \cup \partial_{\text{ext}} W(S \mid B)$ and two vertices are connected with an edge if they are D-connected. From Lemma 10 this graph is connected and from item (b) of Lemma 9 all the elements of B are among its vertices. By eliminating some edges, we turn this graph into a tree. Let us denote t the number of its vertices. From Lemma 13

$$t \leqslant |B| + |\partial_{\text{ext}} W(S \mid B)| \leqslant (1 + d \cdot 4^d) \cdot |B|.$$

For any tree with t vertices there is a path in it with length $2t-2$, which visits all its vertices at least once and returns to the initial vertex. Let us choose such a path and encode it by a sequence of symbols. Every element of $\mathbb{Z}^d$ is D-connected with $3^d - 1$ other elements, so to encode the direction of one step of our path we need a choice of $3^d - 1$ symbols. In addition, we need to encode whether the present vertex belongs to B or not, so the number of choices which it is sufficient to have at every step is $2 \cdot (3^d - 1) < 3^{d+1}$. The length of the coding sequence is

$$2t - 2 \leqslant 2t \leqslant 2 \cdot (1 + d \cdot 4^d) \cdot |B|.$$

Let us pad some more terms to this sequence at the end of it so that the path remains within the same tree to make the length of the sequence become exactly $2 \cdot (1+d \cdot 4^d) \cdot k$, where $k = |B|$. Every such sequence determines B only up to a shift, but due to Lemma 1 at most one shift is a minimal barrier for S. Thus

$$M_k \leqslant C^k, \qquad \text{where} \quad C = (3^{d+1})^{2 \cdot (1+d \cdot 4^d)}. \quad \blacksquare \tag{4}$$

Due to Lemma 14, the estimation in Lemma 3 turns into

$$P_{\text{cut}}(S) \leqslant \sum_{k=k_0}^{\infty} (C \cdot \varepsilon)^k = \frac{(C\varepsilon)^{k_0}}{1 - C\varepsilon} \leqslant (C_d \cdot \varepsilon)^{k_0},$$

where $k_0 = k_0(S)$, C is taken from (4), $C_d = 2C$ and $\varepsilon < \varepsilon_d^* = 1/(2C)$. *Theorem 1 is proved.*

Part II. Cutting Pancakes

Here we apply Theorem 1 to a more special kind of sets to be cut from infinity, but with a more general pattern of percolation. We assume that there is a finite set $\{n_1, \ldots, n_v\} \subset \mathbb{Z}^d$ of non-zero integer vectors called *neighbor vectors* and consider vertex percolation on the oriented graph N, whose set of vertices is $\mathbb{Z}^d$ and an oriented edge goes from v to w if $w - v \in N$, in which case we call w an *N-neighbor* of v. Our approach includes the unoriented percolation as a special case when N has central symmetry and includes the pattern used in Part I as a special case specified by (1). For any $r \geqslant 0$ we call a *cube* the set

$$\Omega_r = \{x \in \mathbb{R}^d : \|x\| \leqslant r\}.$$

Geometrically, Ω_r is a cube with center at the origin and side $2r$, whose edges are parallel to the axes. We call *pancakes* sets denoted and defined as

$$\Theta_{L,r,\rho} = (L \cap \Omega_r) + \Omega_\rho,$$

where L is a linear subspace of $\mathbb{R}^d$. We call L *direction*, $\dim(L)$ *dimension*, r *radius* and ρ *thickness* of the pancake $\Theta_{L,r,\rho}$. Our purpose is to estimate the asymptotic behavior of probability that a *discrete pancake* $\Theta_{L,r,\rho} \cap \mathbb{Z}^d$ is cut from ∞ when radius r tends to ∞ while d, N, L and ρ remain constant, ρ being large enough.

Whenever we use the word "constant" or abbreviation const, we mean a positive number, which does not depend on r. If f and g are two positive functions of the same arguments, $f \asymp g$ means that there is a positive constant C, which we call an *estimation constant*, such that $f \leqslant C \cdot g$ and $g \leqslant C \cdot f$. When we want to mention the estimation constant explicitly, we use the sign $\overset{C}{\asymp}$ with the same meaning.

Theorem 2. For any graph N, where $\{n_1, \ldots, n_\nu\}$ is a finite set of non-zero integer vectors, among which at least two are non-collinear (whence $d \geqslant 2$), and any L a linear subspace of $\mathbb{R}^d$, there are $\varepsilon^* > 0$ and ρ_0 such that for any $\varepsilon \in (0, \varepsilon^*)$ and $\rho \geqslant \rho_0$ there is $C > 0$ such that for all $r \geqslant 0$

$$-\ln P_{\text{cut}}(\Theta_{L, r, \rho} \cap \mathbb{Z}^d) \overset{C}{\asymp} \begin{cases} r^{\dim(L)-1} & \text{if } \{n_1, \ldots, n_\nu\} \subseteq L, \\ r^{\dim(L)} & \text{if } \{n_1, \ldots, n_\nu\} \nsubseteq L. \end{cases}$$

One may compare this theorem with that of ref. 7, which is also formulated in terms of directions of neighbor vectors in oriented percolation.

Proof of Theorem 2. From now on we fix d, N and L satisfying the assumptions of Theorem 2. Notice that since $-\ln(\cdot)$ is a decreasing function, upper estimations in Theorem 2 are based on lower estimations of $P_{\text{cut}}(\Theta_{L, r, \rho} \cap \mathbb{Z}^d)$ and vice versa.

Lemma 15.

(a) For any convex set $S \subset \mathbb{R}^d$ the set $(S + \Omega_{1/2}) \cap \mathbb{Z}^d$ is D-connected.

(b) For any convex set $S \subset \mathbb{R}^d$ and any $\rho \geqslant 1/2$ the set $(S + \Omega_\rho) \cap \mathbb{Z}^d$ is D-connected.

(c) Whenever $\rho \geqslant 1/2$, the set $\Theta_{L, r, \rho} \cap \mathbb{Z}^d$ is D-connected.

Proof of (a). For any $v \in \mathbb{Z}^d$ let us call *unit cube centered at* v and denote cube(v) the set $v + \Omega_{1/2}$. Notice that all unit cubes are closed, that they cover all the space $\mathbb{R}^d$, that their pairwise intersections have volume (d-dimensional measure) zero, that two unit cubes intersect if and only if their centers are D-neighbors and that $(S + \Omega_{1/2}) \cap \mathbb{Z}^d$ is the set of those vertices, whose unit cubes intersect S. Now let us denote $S' = (S + \Omega_{1/2}) \cap \mathbb{Z}^d$, take any $v, w \in S'$ and prove that they are connected with a D-path in S'. Since unit cubes centered at v and w intersect S, we can take some $p \in \text{cube}(v) \cap S$ and $q \in \text{cube}(w) \cap S$ and connect them with a segment. Since S is convex, this segment belongs to S. Since unit cubes with centers in S' cover S, they cover this segment. Since S is limited, S' is finite. So this segment can be cut into a finite sequence of pieces, each belonging to a unit cube of some element of S'. Every two next unit cubes in this sequence intersect, whence their centers are D-neighbors. So we have a D-path in S' connecting v with w. ∎

Proof of (b). Since any Ω_r is convex and $\Omega_a+\Omega_b=\Omega_{a+b}$, we may represent $S+\Omega_\rho=(S+\Omega_{\rho-1/2})+\Omega_{1/2}$, so item (b) follows from item (a).

Proof of (c). It follows from item (b) and definition of $\Theta_{L,r,\rho}$ since $L\cap\Omega_r$ is convex. ▮

Lemma 16. For any $d, L, \rho\geqslant 1/2$ and N specified in Theorem 2 there is $C>0$ such that for all $r\geqslant 0$

$$|\Theta_{L,r,\rho}\cap\mathbb{Z}^d|\overset{C}{\asymp} r^{\dim(L)}.$$

Proof. Estimation from Below. Notice that $\Theta_{L,r,\rho}=\Theta_{L,r,\rho-1/2}+\Omega_{1/2}$. Let us denote

$$S=\Theta_{L,r,\rho}\cap\mathbb{Z}^d=(\Theta_{L,r,\rho-1/2}+\Omega_{1/2})\cap\mathbb{Z}^d.$$

Since unit cubes with centers in S cover $\Theta_{L,r,\rho-1/2}$ and the volume of each of their intersections with $\Theta_{L,r,\rho-1/2}$ does not exceed a constant, their number is not less than a positive constant times volume of $\Theta_{L,r,\rho-1/2}$, which is $\asymp r^{\dim(L)}$.

Estimation from Above. The union of unit cubes with centers in S belongs to $\Theta_{L,r,\rho}+\Omega_{1/2}=\Theta_{L,r,\rho+1/2}$. Pairwise intersections of different unit cubes have volume zero. So their number does not exceed a constant multiplied by the volume of $\Theta_{L,r,\rho+1/2}$, which is $\asymp r^{\dim(L)}$. ▮

Lemma 17. For any $d, L, \rho\geqslant 1/2$ and N specified in Theorem 2 there is $C>0$ such that for all $r\geqslant 0$

$$k_0(\Theta_{L,r,\rho}\cap\mathbb{Z}^d)\overset{C}{\asymp}\begin{cases} r^{\dim(L)-1} & \text{if}\quad \{n_1,\ldots,n_\nu\}\subseteq L,\\ r^{\dim(L)} & \text{if}\quad \{n_1,\ldots,n_\nu\}\nsubseteq L.\end{cases}$$

Proof. Upper Estimations. First let $\{n_1\ldots,n_\nu\}\nsubseteq L$. From Lemma 16, $|\Theta_{L,r,\rho}\cap\mathbb{Z}^d|\asymp r^{\dim(L)}$. Since the set $\Theta_{L,r,\rho}\cap\mathbb{Z}^d$ serves as its own barrier, the upper estimation immediately follows. Now let $\{n_1,\ldots,n_\nu\}\subseteq L$. In this case we use the set

$$S=(L\cap(\Omega_{r+\|N\|}\setminus\Omega_r))+\Omega_\rho,$$

where $\|N\|=\max(\|n\|, n\in N)$. It is evident that $S\cap\mathbb{Z}^d$ is a barrier for $\Theta_{L,r,\rho}\cap\mathbb{Z}^d$ and that cardinality of $S\cap\mathbb{Z}^d$ grows as $\text{const}\cdot r^{\dim(L)-1}$ when $r\to\infty$, whence the upper estimation follows.

Lower Estimations. Let us choose any $n\in\{n_1,\ldots,n_\nu\}$ and draw lines parallel to n through all the elements of $\Theta_{L,r,\rho}\cap\mathbb{Z}^d$. Any barrier for this set must intersect all these lines, whence the number of elements in a barrier cannot be less than the number of these lines. Let us estimate this number in our two cases.

First let $\{n_1,..., n_\nu\} \subseteq L$. From Lemma 16 $|\Theta_{L,r,\rho} \cap \mathbb{Z}^d| \asymp r^{\dim(L)}$. The number of elements of $\Theta_{L,r,\rho} \cap \mathbb{Z}^d$ belonging to one line does not exceed a constant times the diameter of this set and therefore does not exceed $\text{const} \cdot r$. So the number of different parallel lines drawn through all the elements of $\Theta_{L,r,\rho} \cap \mathbb{Z}^d$ cannot be less than the ratio of these numbers, namely $\text{const} \cdot r^{\dim(L)-1}$.

Now let $\{n_1..., n_\nu\} \nsubseteq L$. Let us choose $n \in \{n_1,..., n_\nu\} \setminus L$. In this case the number of elements of $\Theta_{L,r,\rho} \cap \mathbb{Z}^d$ belonging to one line parallel to n does not exceed a constant. Therefore the number of different lines parallel to n drawn through all the elements of $\Theta_{L,r,\rho} \cap \mathbb{Z}^d$ cannot be less than a constant times the cardinality of this set and therefore cannot be less than $\text{const} \cdot r^{\dim(L)}$. ∎

The lower estimation of Theorem 1, item (c) of Lemma 15 and the upper estimations of k_0 provided by Lemma 17 immediately infer the *upper estimations* of Theorem 2.

Now let us prove the *lower estimations* of Theorem 2. For the case (1) they immediately follow from the upper estimation of Theorem 1 and Lemma 17. *Let us consider another special case when there are only two linearly independent neighbor vectors* $n_1, n_2 \in \mathbb{Z}^d$. Now graph N may be disconnected. All its connected components are shifts of N', where N' is the sub-graph of N, whose vertices are linear combinations of n_1, n_2 with integer coefficients, from every site v of which oriented edges go to $v+n_1$, $v+n_2$. Graph N' is isomorphic with the graph N considered in Part I with $d=2$. Also let us denote N'' the minimal linear space containing N', that is the set of linear combinations of n_1, n_2 with real coefficients. Clearly, if $S \subset S' \subset \mathbb{Z}^d$ and S' is cut from ∞, then S is cut from ∞ also, so

$$S \subset S' \subset \mathbb{Z}^d \Rightarrow P_{\text{cut}}(S) \geqslant P_{\text{cut}}(S'). \tag{5}$$

Thus to obtain a lower estimation for $-\ln P_{\text{cut}}(\Theta_{L,r,\rho} \cap \mathbb{Z}^d)$ it is sufficient to obtain an analogous estimation for some subset of $\Theta_{L,r,\rho} \cap \mathbb{Z}^d$.

For any set $S \subset \mathbb{R}^d$ let us call *discrete slices* of S non-empty intersections of S with sets of vertices of connected components of the graph N. To every discrete slice of a set $S \subset \mathbb{R}^d$ there corresponds a *slice* of S, which is the intersection of S with that shift of N'', which contains this discrete slice. For any n_1, n_2 there is a number α such that to every slice there correspond α discrete slices. Evidently, $\Theta_{L,r,\rho} \cap \mathbb{Z}^d$ is cut from ∞ if and only if all its discrete slices are cut from ∞. Since all these events are independent,

$P_{\text{cut}}(\Theta_{L,r,\rho} \cap \mathbb{Z}^d)$ equals the product of probabilities that all its discrete slices are cut from ∞. Since we need an estimation of $P_{\text{cut}}(\Theta_{L,r,\rho} \cap \mathbb{Z}^d)$ from above, we may ignore some discrete slices due to monotonicity (5).

Let us denote $M: N'' \to \mathbb{R}^2$ that linear transformation, which maps n_1 into e_1 and n_2 into e_2. Also we denote $m = \min_{v \in (N'' \setminus \{0\})} \|Mv\|/\|v\|$. Since M is non-degenerate, $m > 0$. Notice that $M(\Omega_r) \supseteq \Omega_{m \cdot r}$ and that

$$M((L \cap \Omega_r) + \Omega_\rho) = ((M(L) \cap M(\Omega_r)) + M(\Omega_\rho)).$$

Therefore $M((L \cap \Omega_r) + \Omega_\rho) \supseteq (L \cap \Omega_{m \cdot r}) + \Omega_{m \cdot \rho}$, that is

$$M(\Theta_{L,r,\rho}) \supseteq \Theta_{L,m \cdot r, m \cdot \rho}.$$

Now let us consider several cases.

First let $\{n_1,..., n_v\} \subset L$, *whence* $N'' \subseteq L$. In this case we may take any $\rho \geqslant 0$. Let us denote

$$C_r = \Theta_{L,r,\rho} \qquad \text{and} \qquad C'_r = \Theta_{L,r/2,\rho/2}.$$

Then from Lemma 16, $|C_r| \asymp |C'_r| \asymp r^{\dim(L)}$. Since the diameter of C'_r is $\asymp r$, the diameter of every discrete slice of C'_r does not exceed $\asymp r$, whence cardinalities of all discrete slices of C'_r do not exceed $\text{const} \cdot r^2$. Therefore the number of discrete slices of C'_r is not less than $\text{const} \cdot r^{\dim(L)-2}$. For every slice of C'_r that slice of C_r, which contains it, contains a disk of radius $\text{const} \cdot r$. Therefore k_0 of the corresponding discrete slice is not less than $\text{const} \cdot r$. The M-image of the intersection of this disk with the corresponding component of N is evidently D-connected, so we can apply Theorem 1 to it. So $-\ln P_{\text{cut}}$ of this discrete slice is not less than $\text{const} \cdot r$ from Theorem 1, so $-\ln$ of the product of these probabilities is not less than $\text{const} \cdot r^{\dim(L)-1}$. Thus in this case $-\ln P_{\text{cut}}(\Theta_{L,r,\rho}) \asymp r^{\dim(L)-1}$.

Now let $\{n_1,..., n_v\} \nsubseteq L$. This case, in its turn, consists of the following two cases.

First let $\dim(N'' \cap L) = 1$. Let us call *good* those slices of C_r, which intersect C'_r. We also call good the corresponding discrete slices. Since the diameter of every good slice is $\asymp r$, cardinalities of good discrete slices are $\asymp r$. Therefore their number is not less than $\text{const} \cdot r^{\dim(L)-1}$. Taking ρ large enough, namely $\rho \geqslant 1/m$, we can assure that the M-images of good discrete slices are D-connected. So we can apply Theorem 1 to these M-images to obtain that $-\ln P_{\text{cut}}$ of every good discrete slice is not less than $\text{const} \cdot r$. Therefore $-\ln$ of the product of these probabilities is not less than $\text{const} \cdot r^{\dim(L)-1}$.

Now let $\dim(N'' \cap L) = 0$. In this case all the slices are uniformly limited, so the number of discrete slices is $\asymp |\Theta_{L,r,\rho}| \asymp r^{\dim(L)}$. Let us denote p the probability that a set consisting of one vertex is cut from ∞. Of course, $p < 1$. Then, from monotonicity (5), for any discrete slice the probability that it is cut from ∞ does not exceed p, whence $-\ln P_{\text{cut}}(\Theta_{L,r,\rho}) \geqslant \text{const} \cdot r^{\dim(L)-1}$.

Now let us prove the lower estimation of Theorem 2 in the general case. Suppose that we have two sets of neighbor vectors, such that the first set belongs to the second. We denote the corresponding probabilities P_{cut} and P'_{cut}. It follows from monotonicity that

$$P'_{\text{cut}}(\Theta_{L,r,\rho} \cap \mathbb{Z}^d) \geqslant P_{\text{cut}}(\Theta_{L,r,\rho} \cap \mathbb{Z}^d).$$

Now take any set $(n_1,..., n_\nu)$ of neighbor vectors and take its subset, which contains only two non-collinear vectors, at least one of which must not belong to L if not all $n_1,..., n_\nu$ belong to L. For this subset the estimation is obtained in the previous case. Now, from monotonicity it follows for the general case also. *Theorem 2 is proved.*

Part III. Large Deviations in Invariant Measures of Some Cellular Automata

Now let us apply our arguments to one class of cellular automata, which are linear operators on the set $\mathscr{M}$ of normed measures on the configuration space $\{0, 1\}^{Z^d}$. We take any finite set $V = \{v_1,..., v_\nu\} \subset \mathbb{Z}^d$, which contains at least two different elements. Then we define a deterministic operator D: $\{0, 1\}^{Z^d} \to \{0, 1\}^{Z^d}$ by the rule: for all $x \in \{0, 1\}^{Z^d}$ and all $v \in \mathbb{Z}^d$

$$(Dx)_v = \min(x_{v+v_1},..., x_{v+v_\nu}).$$

Also for any $\varepsilon \in [0, 1]$ we define one-sided noise R_ε: $\mathscr{M} \to \mathscr{M}$ as follows: when applied to a measure δ_x concentrated in a configuration $x = (x_v)$, it produces a product measure $R_\varepsilon \delta_x$, in which the vth component equals 1 with a probability 1 if $x_v = 1$ and with a probability ε if $x_v = 0$. Let us denote

$$\mu = \lim_{t \to \infty} (R_\varepsilon D)^t \, \delta_0,$$

where δ_0 is the measure concentrated in the configuration "all zeros." Existence of this limit is well-known, it follows from monotonicity. For any set $S \subset Z^d$ we denote $\mathbb{1}(S)$ the cylinder set "all components in S are ones."

Theorem 3. For any dimension $d \geqslant 1$, any linear subspace $L \subseteq \mathbb{R}^d$ and any set $V \subset \mathbb{Z}^d$ containing at least two different elements there is $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$ there is $C > 0$ such that for all $r > 1$

$$-\ln \mu(\mathbb{1}(\Theta_{L,r,\rho} \cap \mathbb{Z}^d)) \overset{C}{\asymp} r^{\dim(L)}.$$

In the case $L = \mathbb{R}^d$ this estimation makes a contrast with estimations of analogous quantities in ref. 8, which describe a class of cellular automata, for whose invariant measures $-\ln \mu(\mathbb{1}(\Omega_r \cap \mathbb{Z}^d)) \asymp r^c$, where $c < d$.

Proof of Theorem 3. It is sufficient to prove the analog of inequality in Theorem 3 for measures $(R_\varepsilon D)^t \delta_0$ uniformly in t:

$$-\ln \mu_t(\mathbb{1}(\Theta_{L,r,\rho} \cap \mathbb{Z}^d)) \asymp r^{\dim(L)}, \qquad \text{where} \quad \mu_t = (R_\varepsilon D)^t \delta_0. \tag{6}$$

We can represent the measure μ_t as induced by auxiliary i.i.d. variables $\eta_{v,s}$, everyone of which equals

$$\eta_{v,s} = \begin{cases} 1 & \text{with probability } \varepsilon, \\ 0 & \text{with probability } 1-\varepsilon \end{cases}$$

and a map defined inductively as follows:

$$\begin{cases} x_{v,0} = 0 & \text{for all} \quad v \in \mathbb{Z}^d, \\ x_{v,s} = \max(\eta_{v,s}, \min(x_{v+v_1, s-1}, \ldots, x_{v+v_n, s-1})) & \text{for all } v \in \mathbb{Z}^d, \ s = 1, 2, \ldots. \end{cases}$$

The upper estimation in (6) is trivial, because the event $\mathbb{1}(\Theta_{L,r,\rho} \cap \mathbb{Z}^d)$ at time t will be assured as soon as $\eta_{v,t} = 1$ for all $v \in \Theta_{L,r,\rho} \cap \mathbb{Z}^d$. The probability that this happens is ε^k, where $k = |\Theta_{L,r,\rho} \cap \mathbb{Z}^d|$. Since $k \asymp r^{\dim(L)}$, this immediately implies the upper estimation in (6). To prove *the lower estimation* in (6), let us observe that we are dealing with percolation in a $(d+1)$-dimensional space with neighbor vectors $\{n_1, \ldots, n_\nu\}$, where

$$n_i = (v_i, -1) \qquad \text{for all} \quad i = 1, \ldots, \nu, \tag{7}$$

any vertex (w, s) with $s > 0$ being closed if and only if $\eta_{w,s} = 1$. Then $x(v, t) = 0$ if and only if there is an open path from the point (v, t) to the initial layer. Now we can interpret $\mu_t(\mathbb{1}(\Theta_{L,r,\rho} \cap \mathbb{Z}^d))$ as the probability that the initial layer $t = 0$ is not reachable from the set

$$(\Theta_{L,r,\rho} \cap \mathbb{Z}^d) + (0, t),$$

which is a shift of $\Theta_{L,r,\rho} \cap \mathbb{Z}^d$ by the vector $(0, t)$. If the initial layer is not reachable from this set, then ∞ certainly is not reachable. From uniformity we may consider $\Theta_{L,r,\rho} \cap \mathbb{Z}^d$ unshifted. Thus:

$$\mu_t(\mathbb{1}(\Theta_{L,r,\rho} \cap \mathbb{Z}^d)) \leqslant P_{\text{cut}}(\Theta_{L,r,\rho} \cap \mathbb{Z}^d).$$

Here $P_{\text{cut}}(\Theta_{L,r,\rho} \cap \mathbb{Z}^d)$ denotes the probability that the set $\Theta_{L,r,\rho} \cap \mathbb{Z}^d$ is cut from ∞ in the graph N, where neighbor vectors are defined by (7). Although $\Theta_{L,r,\rho} \cap \mathbb{Z}^d$, which was defined in a d-dimensional space, is not $\Theta_{L,r,\rho} \cap \mathbb{Z}^d$ in the (d+1)-dimensional space, it still has all the properties we need: it is D-connected and its k_0 equals its cardinality, which grows as $\asymp r^{\dim(L)}$. So we can apply Theorem 1 to obtain the lower estimation of (6). ∎

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